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Modelling of the stiffness of elastic body

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Abstract

It is necessary for designing vibration-isolation systems to know the components' static stiffness, dynamic stiffness and shock stiffness, which are obtained through experiment at present. If the stiffness model of (viscous) elastic body is set-up, the essence of stiffness will be clearer and the experiment simpler. This paper presents a new method for modelling the stiffness of elastic body with viscoelastic theory. The parameters of the model set-up by using this method can be determined easily and present the characteristics of the elastic body's static stiffness, dynamic stiffness and shock stiffness.

1. Introduction

The stiffness of elastic body is defined as partial differential of the acting force on the elastic body to its corresponding displacement [1]

$$k = \frac{\partial F}{\partial x}. \tag{1}$$

If the displacement depends only on the acting force and its time history,

$$x(t) = f\left(F(t), \int_0^t F(\tau)h_1(t-\tau)\,\mathrm{d}\tau, \int_0^t \int_0^t F(\tau)h_2(t-\tau_1, t-\tau_2)\,\mathrm{d}\tau_1\,\mathrm{d}\tau_2, \ldots\right), \tag{2}$$

then the stiffness will be the function of the displacement and its derivative:

$$k = \frac{\partial F}{\partial x} = k(x, \dot{x}, \ddot{x}, \ldots). \tag{3}$$

In engineering, different forms of the stiffness are needed (such as the static stiffness, dynamic stiffness and shock stiffness, and they correspond to the static load, dynamic load and shock

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load). For most of the elastic materials, the value of the static stiffness is the smallest; the shock stiffness is the greatest and the dynamic stiffness lies between them.

Rubber is a kind of viscoelastic material and is widely used in engineering. The differences between its stiffnesses (static stiffness, dynamic stiffness and shock stiffness) are large, and these differences cannot be neglected, especially in the antishock design process. A uniform model of the static stiffness, dynamic stiffness and shock stiffness can be set-up using the theory of viscoelasticity [4–7]. The standard three-parameter solid model describes qualitatively the stiffness with respect to a very limited range of frequency through its inner retardation mechanisms. In order to describe the stiffness frequency characteristics in a broader range of frequency, more retardation mechanisms that work at different frequencies should be considered in the model. This model is the so-called general Maxwell model consisting of several Maxwell bodies, which suggests a high order differential equation describing the relationship of the force and the strain of the material. The more bodies are used, the more accurate the fitting is and known parameters are needed. This model is easy to understand and its parameters are easy to get.

2. Static stiffness, dynamic stiffness and shock stiffness

The stiffness defined by Eq. (3) cannot be obtained easily and is inconvenient to use in engineering. In vibration engineering, the stiffness can be defined by the displacement response to a sine force excitation, i.e., exerting a force $F(t) = F\mathrm{e}^{\mathrm{j}\omega t}$ (whose amplitude is F and frequency ω) on the elastic body, generally which is a periodic function of frequency ω generally, $X(t) = X\mathrm{e}^{\mathrm{j}\omega t+\varphi} = X^{*}\mathrm{e}^{\mathrm{j}\omega t}$ yields the steady displacement response. Thus, the complex stiffness is defined by

$$k^{*} = \frac{F}{X^{*}} = k^{*}(\omega, X). \tag{4}$$

If the system is linear, i.e., if the stiffness is independent of the amplitude of force, i.e., if $k^{*} = k^{*}(\omega)$, then the complex stiffness can be determined by amplitude $k = F/X$, and phase angle φ, both of which are functions of frequency ω. The static stiffness, dynamic stiffness and shock stiffness can all be obtained from complex stiffness. This definition can be generalized to common springs, dampers and their combinations. Note that the above definition is based on the assumption that the initial displacement, velocity and acceleration are all zero before loading.

If the constitutive relationship of the material is independent of the strain velocity (and this in fact is true for some materials like steel spring with independent of loading speed) then stiffness is independent of the frequency. If the constitutive relationship of the material dependent on the strain velocity (and this is in fact is true for some materials like rubber, whose displacement is dependent on the loading speed) then $k(0)$ is its static stiffness and $k(\infty)$ its shock stiffness. In engineering, ω can be taken as the typical frequency of the excitation force or the natural frequency of the system (because the response to it is dominant), hence, $k(\omega)$ is the so-called dynamic stiffness.

For the materials whose constitutive relationships are independent of strain velocity, there is no difference among their stiffnesses, static stiffness, dynamic stiffness and shock stiffness. For those materials whose constitutive relationships are dependent on the strain velocity, in general, the static stiffness is the smallest, the shock stiffness is the greatest and the dynamic stiffness lies between them. For the elastic bodies the static stiffness is greater than zero and the shock stiffness is somewhat limited.

3. Standard three-parameter solid model

In vibration engineering, it is assumed that the stress in the material will never reach the yielding point, i.e., plastic deformation of the elastic body will never happen. Thus, viscoelasicity theory can be used to model its stiffness [2].

As for Maxwell model (Fig. 1(a))

$$x = \frac{F_0}{k}\sin\omega t - \frac{F_0}{c\omega}\cos\omega t, \tag{5}$$

$$k(\omega) = \frac{kc\omega}{\sqrt{k^2 + c^2\omega^2}}. \tag{6}$$

The static stiffness: $k(0) = 0$, the shock stiffness: $k(\infty) = k$.

As for Voigt model (Fig. 1(b))

$$x = k\cos\omega t + c\omega\sin\omega t, \tag{7}$$

$$k(\omega) = \sqrt{k^2 + \omega^2 c^2}. \tag{8}$$

The static stiffness: $k(0) = k$, the shock stiffness: $k(\infty) = \infty$.

The stiffnesses mentioned above disagree with the facts and they are unable to present the stiffness of the viscoelastic body correctly.

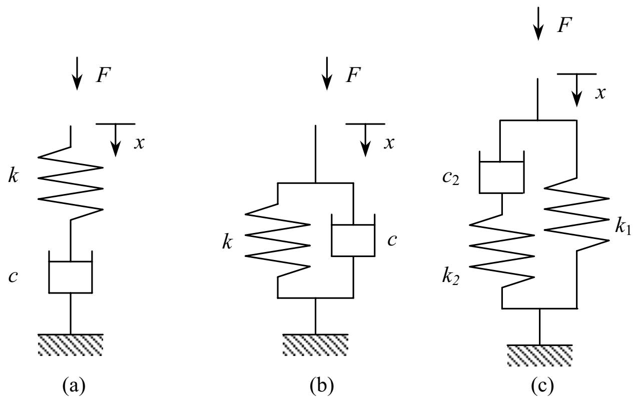

Fig. 1. Three different models of viscoelastic body: (a) Maxwell model, (b) Voigt model, and (c) standard solid model.

As for standard solid model (Fig. 1(c))

$$\frac{dx}{dt} + \frac{k_1 k_2}{c_2(k_1 + k_2)} x = \frac{k_2}{c_2(k_1 + k_2)} F + \frac{1}{(k_1 + k_2)} \frac{dF}{dt}, \tag{9}$$

$$k(\omega) = \sqrt{\frac{k_1^2 k_2^2 + (k_1 + k_2)^2 c_2^2 \omega^2}{k_2^2 + \omega^2 c_2^2}}, \tag{10}$$

$$\tan\delta = \frac{k_2^2 c_2 \omega}{k_1 k_2^2 + (k_1 + k_2) c_2^2 \omega^2}, \tag{11}$$

the static stiffness $k(0) = k_1$, the shock stiffness $k(\infty) = k_1 + k_2$.

This agrees with the fact in form.

Eq. (10) can be given in terms of the static stiffness and the shock stiffness

$$k_1 = k(0), \quad k_2 = k(\infty) - k(0), \quad k(\omega) = \sqrt{\frac{k^2(0)(k(\infty) - k(0))^2 + k^2(\infty) c_2^2 \omega^2}{(k(\infty) - k(0))^2 + c_2^2 \omega^2}}. \tag{12}$$

$\tan\delta$ has its maximum when $\omega = \omega_n$

$$\text{tg}\,\delta_{\max} = \frac{k_2}{\sqrt{k_1^2 + k_1 k_2}} = \frac{k(\infty) - k(0)}{\sqrt{k(0) k(\infty)}}, \tag{13}$$

$$k(\omega_n) = \sqrt{k_1(k_1 + k_2)} = \sqrt{k(\infty) k(0)}. \tag{14}$$

The curves in Fig. 2 are the stiffness- and phase-frequency characteristics of the model. The complex stiffness of the model can be divided into three sections in the frequency domain: in the lower frequency range, the model embodies the static stiffness almost without damp; in the higher frequency range, the model embodies the shock stiffness almost without damp; in the middle frequency range, the stiffness is almost linear with logarithm frequency, and the damp has a maximum and the frequency corresponding to this maximum is one of the model's natural characteristics and this implies the acting frequency of the retardation mechanism is dependent on the strain velocity of the material. The frequency can be considered to be the characteristic

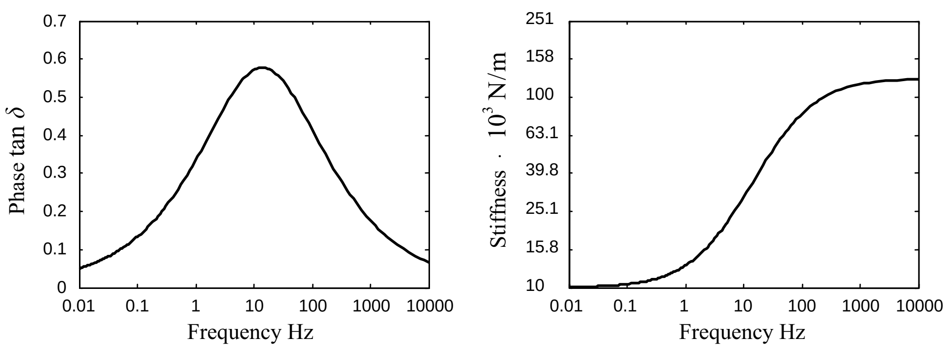

Fig. 2. Stiffness and phase frequency of the standard three-parameter solid model.

frequency of the retardation mechanism. Less than this frequency, the retardation mechanism works very slowly and makes no difference; above this frequency, it is too late to work and makes no difference. Only at the characteristic frequency can the function of retardation mechanism be the greatest.

The determination of parameters can be carried out fitted by the approximate method as follows: First, take the stiffness at the lower frequency as constant, i.e., the static stiffness $k(0)$, and take the stiffness at higher frequency as constant, i.e., the shock stiffness $k(\infty)$ $(k(0) \leqslant k(\infty))$, so that the stiffness parameters k_1 and k_2 are determined. Second, determine the characteristic frequency, that is, determine ω_n when the stiffness is equal to $\sqrt{k(\infty)k(0)}$. Then the damp $c = (k(\infty) - k(0))\sqrt{k(0)/k(\infty)}/\omega_n$. The static stiffness and the shock stiffness can be determined using the least-squares method to make the error the least.

4. General Maxwell body

For actual viscoelastic body, the range of the frequency affecting the stiffness is wide. Hence, it is impossible there to be only one retardation mechanism in the range. There should be a series of retardation mechanisms with different characteristic frequencies and the associated weight factors must be different [3].

The standard three-parameter solid model can be generalized to the general Maxwell model (as shown in Fig. 3) consisting of a series of Maxwell bodies connected in parallel. Every Maxwell body represents a retardation mechanism with certain characteristic frequency. Assuming the characteristic frequency of the 0th Maxwell body to be zero, the model can be fit to represent a kind of solid. Note the characteristic frequency of the whole model is not zero.

For a single Maxwell element, the phase changes greatly around the characteristic frequency, while it remains unchanged at other frequencies. Thus, every Maxwell element can be considered alone. In a certain frequency range, only the element whose retardation mechanism's characteristic frequency matches can work while the others have no function. The damp of the

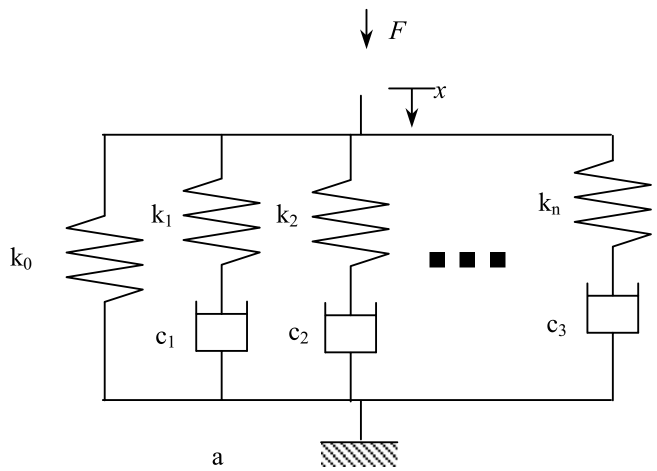

Fig. 3. General Maxwell model.

element whose characteristic frequency is in the lower range is very large, and the element can be considered as a short circuit. That is, the spring is connected with the two ends directly and its stiffness appears as static stiffness. The damp of the element whose characteristic frequency is in the higher range is very small, and the element can be considered as an open circuit. That is, the element contributes nothing to the stiffness. The static stiffness of this frequency range is the outcome of springs connected in parallel whose characteristic frequencies are lower than the frequency. The shock stiffness is determined by the spring of the element whose characteristic frequency matches. The element whose characteristic frequency is higher than the frequency of this section can be neglected.

The number of elements is determined based on the frequency section interested. It is very easy to extend such a model to a broader frequency range: connecting the elements whose characteristic frequencies are higher in parallel will be fit for higher frequency section; for lower frequency section, it is necessary to divide the 0th element into the element with lower characteristic frequency and the spring (the new 0th element) connected in parallel. The sum of the element's elastic parameter and the spring's stiffness is supposed to be equal to the stiffness of the spring of the former 0th element. In other words, the stiffness of the higher frequency section should remain unchanged.

5. Determination of the parameters

Let the parameters of the ith Maxwell body be k_i, c_i, $(i = 0\ldots n, c_0 = \infty)$. The stiffness $k(\omega)$ related to the known frequency can be determined as follows:

(a) Determining the 0th element

$$k_0 = k(0), \quad c_0 = 0.$$

(b) Determining the next element

Assuming $k(\omega)$ changes greatly along with frequency from the very beginning, its characteristic frequency is ω_i, the corresponding stiffness of the upper limit of this frequency section is $k(\omega_i^*)$. Thus

$$k_i = k(\omega_i^*) - k_{i-1}, \quad c_i = \frac{k_i}{\omega_i}\sqrt{\frac{k_0 + k_1 + \cdots + k_{i-1} + k_i}{k_0 + k_1 + \cdots + k_{i-1}}}. \tag{15}$$

(c) Repeating the procedure (b) till $k(\omega)$ hardly changes along with frequency or exceeds the scope of interest.

6. Examples

Fig. 4 shows the stiffness–frequency characteristics of an elastic body with a broad range of frequency. A general Maxwell body with three elements is used to represent the elastic body's viscoelasticity, i.e., the model of the material is simplified as two retardation mechanisms. According to the method introduced above, the parameters of each element can be obtained. The 0th element represents the static stiffness, $k_0 = 11.22 \times 10^3$ N/m, $c_0 = 0$ N s/m; the first element

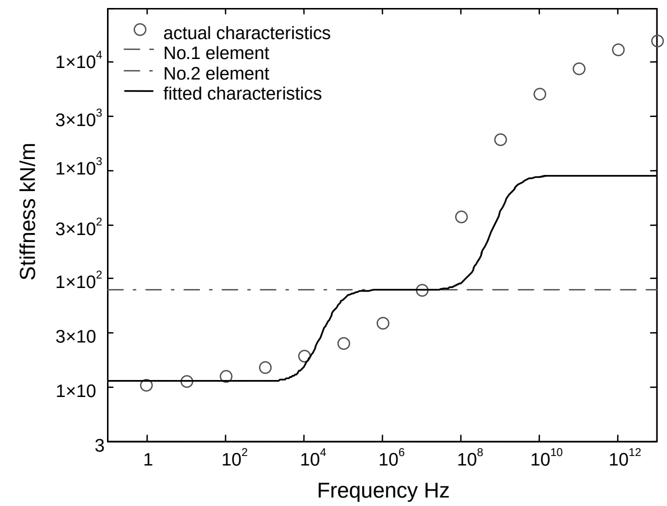

Fig. 4. Stiffness–frequency characteristic of general Maxwell with three elements.

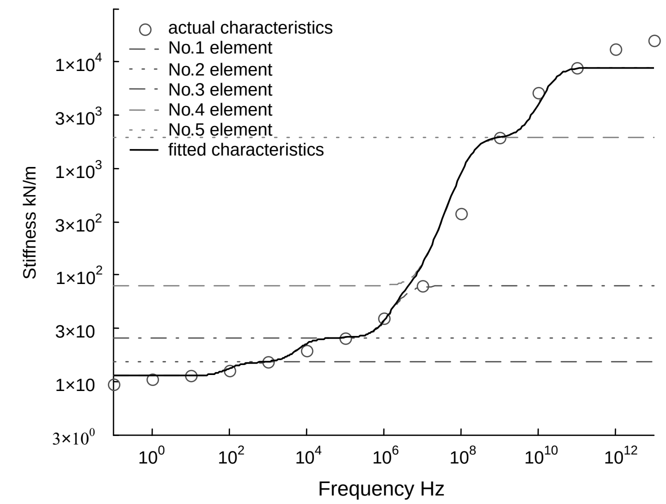

Fig. 5. Stiffness–frequency characteristic of general Maxwell with six elements.

represents the retardation mechanism whose working frequency is in the range 10–10^5 Hz, $\omega_1 = 1.99 \times 10^3$ Hz, $k_1 = 67.65 \times 10^3$ N/m, $c_1 = 0.9011$ N s/m, the second element, has working frequency in the range 10^5–10^{11} Hz, $\omega_2 = 2.51 \times 10^9$ Hz, $k_2 = 8.834 \times 10^6$ N/m, $c_2 = 0.435 \times 10^{-3}$ N s/m. There are five parameters totally and the fitting results are shown in Fig. 4.

The elastic body can also be described as a six-element general Maxwell body (five retardation mechanisms) to represent its viscoelasticity. The parameters of each element are: the 0th element representing the static stiffness, $k_0 = 11.22 \times 10^3$ N/m, $c_0 = 0$ N s/m, the first element representing the retardation mechanism working between 10–10^3 Hz: $\omega_1 = 126.9$ Hz, $k_1 = 3.92 \times 10^3$ N/m,

$c_1 = 35.88\,\mathrm{N\,s/m}$, the second element: 10^3–10^5 Hz, $\omega_1 = 10.1 \times 10^3$ Hz, $k_1 = 10.56 \times 10^3\,\mathrm{N/m}$, $c_1 = 1.376\,\mathrm{N\,s/m}$, the third element: 10^5–10^7 Hz, $\omega_1 = 5.01 \times 10^6$ Hz, $k_1 = 53.16 \times 10^3\,\mathrm{N/m}$, $c_1 = 18.6 \times 10^{-3}\,\mathrm{N\,s/m}$, the fourth element: 10^7 to 10^9 Hz, $\omega_1 = 1.01 \times 10^8$ Hz, $k_1 = 1.871 \times 10^6\,\mathrm{N/m}$, $c_1 = 2.1 \times 10^3\,\mathrm{N\,s/m}$, the fifth element: 10^9–10^{11} Hz, $\omega_1 = 4.98 \times 10^9$ Hz, $k_1 = 6.963 \times 10^9\,\mathrm{N/m}$, $c_1 = 0.297 \times 10^{-3}\,\mathrm{N\,s/m}$. There are totally 11 parameters and the final fitting results are shown in Fig. 5.

The fitting results suggest that the general Maxwell body reflects the relationship of the stiffness and the frequency of the viscoelasticity body on the whole. More elements are used for more accurate fitting, more parameters are needed. In this example, the damping coefficient of each element is almost the same because the relationships of stiffness and frequency of the viscoelastic body are linear. There are more retardation mechanisms in six-element model, and the damping coefficient of each element is smaller accordingly.

7. Conclusion

Maxwell body can be used to describe the relationship between stiffness and frequency of viscoelastic body similar to rubber, with its static stiffness zero, whose nature is liquid with great viscosity. For sulfured rubber with some static stiffness, connecting in parallel a spring without damping to a Maxwell body can make a standard solid model. There is only one retardation mechanism in the standard solid model, which causes a very limited scope of the frequency related to the stiffness and is not enough to describe practical materials. If several Maxwell bodies are connected in parallel to the standard solid body, i.e., if the number of the retardation mechanisms which work at different frequencies are increasing the stiffness frequency characteristics in a wider scope of frequencies can be obtained.

The method using several Maxwell bodies connected in parallel to describe the stiffness frequency characteristics of the viscoelastic body in a broad frequency scope has a clear physical concepts and so are the fitting parameters. Hence, this method is convenient to use. The deficiency of this procedure for higher order model is that the parameters needed to be determined are in this case many, and a lot more data is needed.

Appendix A. Nomenclature

c	damping
c_i	ith element's damping
F	force
k	stiffness
k^*	complex stiffness
k_i	ith element's stiffness
$k(0)$	static stiffness
$k(\omega)$	dynamic stiffness
$k(\infty)$	shock stiffness
t	time

x displacement
δ phase angle
ω angular frequency
ω_i ith element's characteristic frequency

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